PMM U.S.S.R.,Vol.49,No.4,pp.399-404,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

SUFFICIENT CONDITIONS OF FINITENESS OF THE PURSUIT TIME*

B.N. PSHENICHNYI and N.B. SHISHKINA

A new effective way of solving the problem of pursuit which includes the case of non-linear differential equations is proposed. The controlling action of the pursuer is designed according to the position of the game. Ideas previously published in /1-6/ are developed.

1. The differential game is specified in Euclidean space R^n. Here $U \subset R^n$ and $V \subset R^n$

$$z^{\cdot} = f(t, z, u, v),\ z \in R^n,\ u \in U,\ v \in V \tag{1.1}$$

are non-empty compacta, and the dot denotes differentiation with respect to time t. The vector function f is continuous over the set of variables that satisfy the Lipschitz condition with respect to z and can be represented in the form of the sum $f(t, z, u, v) = f_1(t, z, u) + f_2(t, z, v)$. The set $f_1(t, z, U)$ is convex for any values of the variables t and z. The terminal set has the form $M = M_0 + K$, where M_0 is a convex compactum in R^n, and K is a closed convex cone in R^n. The game is considered finished from the initial position t°, z°, if at some instant of time $t > t^\circ$ we have $z(t) \in M$.

Definition 1. Any measureable function $v(t)$, $t \geqslant t^\circ$ with values in V is called the strategy of the pursued in the game (1.1).

Definition 2. Any upper semicontinuous multivalued mapping $U(z)$ from R^n into 2^U, where 2^U is the set of all subsets of compactum U, is called the pursuer's strategy.

We say that the differential game (1.1) may be completed from a given initial position t°, z°, if a strategy of the pursuer $U(z)$ exists such that for any strategy of the pursued $v(t)$ the solution of the differential inclusion

$$z^{\cdot} \in f(t, z, U(z), v(t))$$

reaches the set M in finite time.

2. Let us introduce some notation and prove some ancilliary statements from the theory of convex analysis.

We set for $x \in M$

$$D_x = \{z:\ z = x + \gamma(m - x),\ \gamma > 0,\ m \in M\}$$

For $z \in D_x$ we have the function

$$\lambda(z) = \max\{\lambda > 0:\ x + \lambda^{-1}(z - x) \in M\} \tag{2.1}$$

By virtue of closure of M

$$m(z) = x + \lambda^{-1}(z)(z - x) \in M\} \tag{2.2}$$

Lemma 1. The functions $\lambda(z)$ and $m(z)$ are directionally differentiable and satisfy the Lipschitz condition inside the region of definition.

Proof. From the definition of the function $\lambda(z)$ we have

$$z = x + \lambda(z)(m(z) - x) \tag{2.3}$$

For any $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$ such that $\gamma_1 + \gamma_2 = 1$ and $z_1, z_2 \in D_x$ the equation

$$\gamma_1 z_1 + \gamma_2 z_2 = x + \Lambda\left(\frac{\gamma_1\lambda(z_1)}{\Lambda} m(z_1) + \frac{\gamma_2\lambda(z_2)}{\Lambda} m(z_2) - x\right) \tag{2.4}$$
$$\Lambda = \gamma_1\lambda(z_1) + \gamma_2\lambda(z_2)$$

holds. Because of the definition of the function $\lambda(z)$, the convexity of the set M, and Eq. (2.4), we have $\lambda(\gamma_1 z_1 + \gamma_2 z_2) \geqslant \Lambda$, i.e. that the function $\lambda(z)$ is concave on D_x. From the concavity of $\lambda(z)$ it follows that the properties defined by the lemma are valid for it /7/. This implies that similar properties hold for $m(z)$.

Consider the set

$$N = \{\psi:\ \|\psi\| = 1,\ W_M(\psi) < +\infty\}$$

where $W_M(\psi)$ is the support function of set M. With the assumptions made regarding the set M, the set N is closed, and the function $W_M(\psi)$ is continuous on it. This enables us to define the function

---

*Prikl.Matem.Mekhan.,49,4,517-523,1985

$$F(z, \lambda) = \max_{\psi \in N} \{(\psi, z - x) + \lambda [(\psi, x) - W_M(\psi)]\} =$$
$$\max_{\psi \in N} \lambda \{(\psi, x + \lambda^{-1}(z - x)) - W_M(\psi)\}$$

We put

$$K_x = \mathrm{con}\,\{M - x\} = \{z:\ z = \gamma (m - x),\ \gamma > 0,\ m \in M\}$$

and denote the coneconjugate to the cone K_x by K_x^*.

Lemma 2. The function $F(z, \lambda)$ is differentiable with respect to the direction z^*, λ^*, and

$$F'(z, \lambda(z), z^*, \lambda^*) = \max_{\psi \in K^*_{m(z)}} \{-(\psi, z^*) + \lambda^* (\psi, m(z) - x)\} \tag{2.5}$$

$$F'(z, \lambda(z), 0, 1) \geqslant 0$$

Proof. The function $F(z, \lambda)$ is known to be convex with respect to z and λ /8/. It is consequently differentiable, and its derivative with respect to the direction z^*, λ^* is calculated by the formula

$$F'(z, \lambda, z^*, \lambda^*) = \max_{\psi \in N(z, \lambda)} \{(\psi, z^*) + \lambda^* ((\psi, x) - W_M(\psi))\}$$
$$N(z, \lambda) = \{\psi:\ \psi \in N,\ (\psi, z - x) + \lambda [(\psi, x) - W_M(\psi)] = F(z, \lambda)\}$$

We know that $m \in M$, if and only if $(\psi, m) \leqslant W_M(\psi)$ for all $\psi, \|\psi\| = 1$. Hence, taking into account the definition of the functions $F(z, \lambda)$ and $\lambda(z)$, we obtain $F(z, \lambda(z)) = 0$, i.e. $\psi \in N(z, \lambda(z))$ means that $\lambda(z)((\psi, m(z)) - W_M(\psi)) = 0$ or $(\psi, m - m(z)) \leqslant 0$ for all $m \in M$. Thus, taking into account the definition of the conjugate cone, we obtain $-\psi \in K^*_{m(z)}$. Besides, since $\psi \in N$, we have $N(z, \lambda(z)) = (-K^*_{m(z)}) \cap N$. The first part of the lemma is proved.

Using the definition of the directional derivative, we obtain

$$F'(z, \lambda(z);\ 0, 1) = \lim_{t \downarrow 0} \frac{F(z, \lambda + t) - F(z, \lambda(z))}{t} \geqslant 0$$

since $F(z, \lambda(z)) = 0$, and $F(z, \lambda(z) + t) > 0,\ t > 0$.

Note that, if the set M has a smooth boundary, then by definition, the set $K^*_{m(z)} \cap B$, where B is the unit sphere in R^n, consists of the unique point $T(m(z))$ which is the inner unit normal to the surface continuously dependent on that point.

Lemma 3. Let the set M have a smooth boundary. Then in the region where the following inequalities are satisfied:

$$S(z) > 0,\ S(z) = (T(m(z)),\ m(z) - x) \tag{2.6}$$

the function $\lambda(z)$ is differentiable and its derivative with respect to the direction z^* is determined by the formula

$$\lambda'(z, z^*) = (T(m(z)), z^*)\ S(z) \tag{2.7}$$

Proof. In this case Eq.(2.5) takes the form

$$F'(z, \lambda(z), z^*, \lambda^*) = -(T(m(z)), z^*) + \lambda^* S(z) \tag{2.8}$$

Since $\lambda(z)$ is directionally differentiable, we have

$$\lambda(z + tz^*) = \lambda(z) + t\lambda'(z, z^*) + o(t)$$

Differentiating now the relation $F(z + tz^*, \lambda + t\lambda^*) = 0$, we obtain by virtue of (2.8) lemma (2.7).

Lemma 4. For any point $z \neq x,\ z = x + \lambda(m_0 - x)$ where $m_0 \in \mathrm{int}\, M$ ($\mathrm{int}\, M$ is the interior of M) the inequality $(\psi, m(z) - x) > 0.\ \forall \psi \in K^*_{m(z)} \cap B$ is satisfied.

Proof. Since $m_0 \in \mathrm{int}\, M$, hence $m_0 - \varepsilon y \in M$ for some $\varepsilon > 0$ and any $y, \|y\| \leqslant 1$. Besides, for $\psi \in K^*_{m(z)}$ the inequality $(\psi, m) \geqslant (\psi, m(z)),\ \forall m \in M$ holds. Consequently, $(\psi, m) - \varepsilon(\psi, y) \geqslant (\psi, m(z))$. Substituting into this $m_0 = x + \lambda^{-1}(z - x),\ m(z) = x + \lambda^{-1}(z)(z - x)$, we obtain

$$(\lambda^{-1} - \lambda^{-1}(z))\ (\psi, z - x) \geqslant \varepsilon(\psi, y) \tag{2.9}$$

It can be shown that $\lambda < \lambda(z)$. Taking the maximum of the right side of (2.9) we find that its left side is not less than $\varepsilon\|\psi\| > 0$. Hence $(\psi, m(z) - x) = \lambda(z) \times (\psi, z - x) > 0$, which it was required to prove.

Lemma 5. For any $\varepsilon > 0$ the ε-neighbourhood of the convex set M has a smooth boundary.

Proof. Let x be the boundary point of the set M_ε. Then a vector $\psi, \|\psi\| \neq 0$ exists such that $(\psi, m - x) \geqslant 0$ for any $m \in M$. It is required to prove that the normal at that point is unique and continuously dependent on the point x.

Let now $M_\varepsilon = M + \varepsilon S_1$, where S_1 is the unit sphere in R^n. It can be shown that $W_{M_\varepsilon}(\psi) = W_M(\psi) + \varepsilon\|\psi\|$. Let us assume that

$$F_\varepsilon(m) = \max_{\|\psi\|=1} \{(\psi, m) - W_M(\psi) - \varepsilon\|\psi\|\} \tag{2.10}$$

Now we have $m \in M_\varepsilon$, if and only if $F_\varepsilon(m) \leqslant 0$, and the boundary of M_ε is defined by the equation $F_\varepsilon(x) = 0$, i.e. $F_\varepsilon(m) \leqslant F_\varepsilon(x)$, $m \in M_\varepsilon$ or $(-\psi, m - x) \geqslant 0$. Thus, $-\psi \in K_x^*$ and $\|\psi\| = 1$. Hence $-\psi$ is the normal to M_ε at the point x.

Carrying out the calculations in reverse order, we find that if $-\psi$ is the unit normal at x, then $F_\varepsilon(x) = 0$.

Thus the set of normals to M_ε at the point x coincides with the set of those ψ, taken with the negative sign, at which a maximum is reached in (2.10). But by virtue of the strict concavity of the function $(\psi, x) - W_M(\psi) - \varepsilon\|\psi\|$, the maximum of (2.10) is reached at a single point. This proves that M_ε has at each boundary point a unique normal $T_\varepsilon(x)$. It is shown in /8/ that if the maximum in (2.10) is reached at a single point $\psi(x)$, then $\psi(x)$ depends continuously on x. Hence, $T_\varepsilon(x) = -\psi(x)$ also depends continuously on x, which it was required to prove.

Lemma 6. If the set M has a smooth boundary $x \in M$ and the point $m_0 \in M$ is such that $\lambda(m_0) = 1$, then $T(m_0) \in K_x^* \cap B$.

Proof. Let the conditions of lemma be satisfied. Then $x + \lambda^{-1}(m_0 - x) \overline{\in} M$ when $\lambda > 1$ by virtue of the fact that $\lambda(m_0) = 1$ and the definition of the function $\lambda(z)$. Hence M and the set $\{x + \gamma(m_0 - x) : 0 \leqslant \gamma \leqslant 1\}$ do not intersect. This means that a vector ψ, $\|\psi\| = 1$ exists such that

$$(\psi, m) \geqslant (\psi, x + \gamma(m_0 - x)),\ x \in M,\ 0 \leqslant \gamma \leqslant 1$$

Hence, when $\gamma = 0$, we obtain $(\psi, m) \geqslant (\psi, x)$, i.e. $\psi \in K_x^*$. On the other hand, we obtain $(\psi, m) \geqslant (\psi, m_0)$ as $\gamma \to 1$, i.e. $\psi \in K^*_{m_0}$. Since M has a smooth boundary and $\|\psi\| = 1$, we have $\psi = T(m_0)$.

3. The answer to the question of whether it is possible to terminate the pursuit in a finite time is given by the following theorem.

Theorem 1. Let the terminal set M have a non-empty interior and a smooth boundary; a point $m_0 \in \operatorname{int} M$ and a number $\rho > 0$ exist such that for the point $x = z^\circ + \rho(z^\circ - m_0)$ and its corresponding function $\lambda(z)$ (2.1) for any z that satisfies the condition $1 > \lambda(z) \geqslant \lambda(z^\circ)$ and $t \geqslant t^\circ$

$$\min_{v \in V} \max_{u \in U} (T(m(z)), f(t, z, u, v)) \geqslant \delta \geqslant 0 \tag{3.1}$$

(the function $m(z)$ is defined in (2.2)).

Then the differential game (1.1) beginning at instant t° at the point z° can be completed.

Proof. It can be see that $\lambda(m(z)) = 1$. Indeed, by the definition of $\lambda(z)$, we have $m(z) \in M$ and consequently

$$x + \frac{1}{1}(m(z) - x) = m(z) \in M$$

i.e. $\lambda(m(z)) \geqslant 1$, but if $\lambda(m(z)) > 1$, i.e.

$$x + \frac{1}{\gamma}(m(z) - x) = m \in M,\ \gamma > 1$$

then $m(z) = x + \gamma(m - x)$ and, substituting this expression into (2.2), we obtain

$$m = x + \frac{1}{\gamma\lambda(z)}(z - x)$$

This means that $1 \geqslant \gamma$, which contradicts the assumption.

Furthermore, since $m_0 = x + \frac{1+\rho}{\rho}(z^\circ - x)$, hence $\lambda(z^\circ) \geqslant \frac{\rho}{1+\rho} > 0$, besides, since $m_0 \in \operatorname{int} M$, hence according to Lemma 4 we have $S(z) > 0$ (($S(z)$ is defined in (2.6)). Let us now design the strategy of the pursuer. We assume for all z such that $\lambda(z^\circ) \leqslant \lambda(z) < 1$

$$U(z) = \{u \in U : (T(m(z)), f_1(t, z, U))\} = \max_{u \in U} (T(m(z)), f_1(t, z, u))$$

Since $m(z)$ depends continuously on z, and $T(m)$ is also a continuous function, the set $U(z)$ depends semicontinuously from above on z in the region where condition (2.6) is satisfied /8/. The last condition is necessary, since according to Lemma 3 only in that region is the function $\lambda(z)$ continuously differentiable, and the vanishing of the left side of (2.6) indicates that the boundary of the region of definition of the function $\lambda(z)$ is reached.

Thus, in the region where condition (2.6) is satisfied the differential inclusion

$$z^{\cdot} \in f(t, z, U(z), v(t)), \ t \geqslant t^{\circ}, \ z(t^{\circ}) = z^{\circ} \tag{3.2}$$

is defined, and it has solutions that can be continued fairly far.

Let us consider some of these, for instance $z(t)$. According to /9/ the control $u(t)$ corresponds to it and is a measurable function with values in $U(z(t)) \in U$, such that

$$z^{\cdot} = f(t, z, u(t), v(t)) \tag{3.3}$$

Since the function $\lambda(z)$ inside the region defined by (2.6) is continuously differentiable, by Lemma 3

$$d\lambda(z(t))/dt = \lambda'(z, z^{\cdot}(t)) = (T(m(z)), f(t, z, u(t), v(t)))/S(z)$$

Taking into account the conditions of the theorem and the fact that $u(t) \in U(z(t))$, we obtain the inequality

$$d\lambda(z(t))/dt \geqslant \delta / S(z) \tag{3.4}$$

which implies that the function $\lambda(z)$ increases monotonically along the trajectory $z(t)$.

Consider the quantity $S(z)$. By definition, $T(m(z))$ for any point $m \in M$

$$(T(m(z)), m) \geqslant (T(m(z)), m(z)) = (T(m(z)), x) + S(z)$$

Since $T(m(z))$ is a vector of unit length and the points m and x are fixed, the quantity $S(z)$ has an upper limit. Consequently $\lambda(z(t))$ increases at a non-zero rate and reaches unity, provided that at some instant $S(z)$ it does not vanish in (3.4) and it is not possible to continue the trajectory.

It can be shown that this cannot occur.

Let us assume the contrary, i.e. when $t \uparrow t_*$, $z(t) \to z_*$

$$S(z_*) = 0 \tag{3.5}$$

The function $F(z, \lambda)$ is convex and at the point z_* according to (2.8) $\lambda(z_*)$ its derivatives with respect to z and λ are $-(T(m(z_*)), z_*)$ and $S(z)$, respectively. By virtue of the convexity we have

$$F(z(t), \lambda(z(t))) \geqslant F(z_*, \lambda(z_*)) - (T(m(z_*)), z(t) - z_*) + (\lambda(z(t)) - \lambda(z_*)) S(z_*)$$

Taking into account (3.5) and the fact that $F(z_*, \lambda(z_*)) = 0$, we obtain

$$(T(m(z_*)), z_* - z(t)) \leqslant 0$$

It follows from (3.3) that

$$z_* - z(t) = \int_t^{t_*} f\,d\tau, \quad f = f(\tau, z(\tau), u(\tau), v(\tau))$$

Hence

$$0 \geqslant \int_0^{t_*} (T(m(z_*)), f)\,d\tau = \int_t^{t_*} (T(m(z(\tau)), f)\,d\tau + \int_t^{t_*} (T(m(z_*)) - T(m(z(\tau)), f))\,d\tau$$

Owing to the continuous dependence of $T(m)$ on m, the last terms in the derived formula will be $o(t_* - t)$. On the other hand, by virtue of $u(\tau) \in U(z(\tau))$, the selection of $U(z)$, and the assumptions of the theorem, the integrand in the penultimate term is greater than δ. Thus $\delta(t_* - t) + o(t_* - t) \leqslant 0$, $t \uparrow t_*$ which is impossible when t is fairly close to t_*.

Hence in Eq.(3.4) the quantity $S(z)$ has an upper limit and is always non-zero. Hence the quantity $\lambda(z(t))$ increases with time and reduces to unity at a certain finite instant of time. But the condition $\lambda(z(t)) = 1$ is equivalent to the fact that $z(t) = m(z(t)) \in M$.

The game is completed and so is the proof of the theorem.

The corollary of the theorem is the following Theorem 2, generalizing the results obtained in /1–3/.

Theorem 2. Let M be a convex set, $f(t, z, u, v) = f(t, u, v)$. If the equation

$$\rho(m - z^{\circ}) = f(t, u, v), \ \forall v \in V, \ \forall t \geqslant t^{\circ} \tag{3.6}$$

has a solution $m \in M$, $u \in U$ and $\rho \geqslant \rho_* > 0$, the game (1.1) commencing at the instant t° from the point z° can be terminated on the terminal set M_ε (M_ε is the ε-neighbourhood of the set M, and ε is an arbitrarily small positive number).

Proof. Let $\varepsilon > 0$ and the quantities $m_* \in M$, $u_* \in U$ and $\rho^* > \rho_*$ be selected according to Eq.(3.6) for given $t \geqslant t^{\circ}$ and $v \in V$. We assume $x = z^{\circ} + \varphi(z^{\circ} - m_0)$ where m_0 is an arbitrary point of M.

It can be shown that for fairly small $\varphi > 0$ and $\delta > 0$ the point $f(t, u_*, v)$ together with the δ-neighbourhood belongs to the cone

$$K_{x,\varepsilon} = \{\gamma(m + \varepsilon y - x): \ \gamma > 0, \ m \in M, \ \|y\| \leqslant 1\} = \operatorname{con}\{M_\varepsilon - x\}$$

For this it is sufficient to show that the equation

$$\gamma(m + \varepsilon y - z^\circ - \varphi(z^c - m_0)) = f(t, u_*, v) + \delta y_1$$

is solvable. Setting $\gamma = \rho^*$, $m = m_*$ and taking (3.6) into account, we obtain $\rho^* \varepsilon y_1 = \delta y_1 + \rho^* \varphi(z^\circ - m_0)$ or

$$y = \frac{\delta}{\rho^* \varepsilon} y_1 + \frac{\varphi}{\varepsilon}(z^\circ - m_0)$$

If we select

$$\varphi = \frac{\varepsilon}{2\|z^\circ - m_0\|}, \quad \delta = \frac{1}{2}\rho_* \varepsilon$$

then

$$\|y\| \leqslant \frac{\delta}{\rho^* \varepsilon}\|y_1\| + \frac{\varphi}{\varepsilon}\|z^\circ - m_0\| \leqslant \frac{1}{2}\frac{\rho_*}{\rho} + \frac{1}{2} \leqslant 1$$

Thus the equation obtained is actually solvable, and for any $y_1 \in S$ we have $f(t, u_*, v) + \delta y_1 \in K_{x,\varepsilon}$

Then for $\psi \in K^*_{x,\varepsilon}, \|\psi\| = 1$

$$(\psi, f(t, u_*, v)) \geqslant -\delta(\psi, y_1), \|y_1\| \leqslant 1$$

Taking the maximum of the left and right sides with respect to $u \in U$ and $y_1 \in S_1$, respectively, we obtain

$$\max_{u \in U}(\psi, f(t, u, v)) \geqslant \delta, \ \psi \in K_{x,\varepsilon}, \ v \in V \tag{3.7}$$

Taking as the terminal set M_ε, we now apply Theorem 2. It is obvious that $\operatorname{int} M_\varepsilon \neq \varnothing$, and point $m_0 \in M$ which was used to construct the point x, belongs to $\operatorname{int} M_\varepsilon$. Furthermore, by Lemma 5, the surface M_ε is smooth, and by Lemma 6 $T(m(z)) \in K^*_{x,\varepsilon}$. Hence by virtue of (3.7), we have

$$\max_{u \in U}(T(m(z)), f(t, u, v)) \geqslant \delta > 0$$

Thus, all conditions of Theorem 1 are satisfied, and the pursuit can be consequently completed ater a finite time.

Remarks. 1°. Theorem 2 can be similarly proved when the function f also depends on the position of z.

2°. These results can easily be transferred to the case of several pursuers.

Example 1 (simple pursuit). The motion of the object is given by the differential equation

$$z^\cdot = u - v; \ \|u\| \leqslant \alpha, \ \|v\| \leqslant \beta, \ \alpha > 0, \ \beta > 0, \ z \in R^2$$

The terminal set has the form $M = \{z \in R^2 : \|z\| \leqslant \varepsilon\}$.

Let us explain what form condition (3.1) will have in this problem. We have

$$\min_{\|v\| \leqslant \beta}\max_{\|u\| \leqslant \alpha}(T(m(z)), f(t, z, u, v)) = \min_{\|v\| \leqslant \beta}\max_{\|u\| \leqslant \beta}(T(m(z)), u - v) =$$
$$\max_{\|u\| \leqslant \alpha}(T(m(z)), u) + \min_{\|v\| \leqslant \beta}(T(m(z)), -v).$$

The inner unit normals at each boundary point m of the set M are obviously equal to $-m/\varepsilon$. Consequently

$$\max_{\|u\| \leqslant \alpha}(T(m(z)), u) = \alpha, \ \min_{\|v\| \leqslant \beta}(T(m(z)), -v) = -\beta$$

and condition (3.1) takes the form $\alpha - \beta > 0$.

We assume the point m_0 coincides with the origin of coordinates, and the controls of the pursuer at each realized position of the game are selected to be equal to the vector $-m(z)\alpha\varepsilon^{-1}$, where $m(z) = x + \lambda(z)(z - x)$, $x = (1 + \rho)z^\circ$, z is the current position of the game, and $\lambda(z)$ is a function constructed by the method indicated above.

Example 2. The differential game is specified by the equation

$$z^\cdot = \alpha z + u - v, \ \|u\| \leqslant r, \ \|v\| \leqslant s, \ r \geqslant 0, \ s \geqslant 0, \ \alpha > 0, \ z \in R^2$$
$$M = \{z \in R^2 : \|z\| \leqslant \varepsilon\}$$

In this problem condition (3.1) has the form

$$(T(m(z)), z) > (s - r)/\alpha \tag{3.8}$$

We shall show that when the inequality

$$r - s > \alpha s \tag{3.9}$$

is satisfied, condition (3.8) holds for all z such that $1 > \lambda(z) \geqslant \lambda(z^\circ)$.

Let us set $m_0 = (0, 0)$. Then we have $x = (1 + \rho)z^\circ$, $T(m(z)) = -m(z)/\varepsilon$ as in the first example.

Starting with (3.9), we write

$$\left(\frac{r - s}{\alpha} - \|z\|\right)\frac{1}{\varepsilon - \|z\|} > 1$$

Consequently

$$\left(\frac{r-s}{\alpha}-\|x\|\right)\frac{1}{\varepsilon-\|x\|}>\lambda(z)$$

Using the inequality $(m(z), x)\leqslant\|m(z)\|\|x\|=\varepsilon\|x\|$, we have

$$(-m(z)/\varepsilon,\ x+\lambda(z)\ (m(z)-x))>(s-r)/\alpha$$

Using Eq.(2.3) we obtain the required condition (3.8).

The game can, thus be completed from the initial position z° in a finite time, if its parameters are connected by relation (3.9). The controls of the pursuers are constructed as in the preceding example.

REFERENCES

1. PSHENICHNYI B.N., Simple pursuit by several objects. Kibernetika, 3, 1976.
2. PSHENICHNYI B.N. and RAPPOPORT I.S., On the problem of group pursuit. Kibernetika, 6, 1979.
3. PSHENICHNYI B.N., CHIKRII A.A. and RAPPOPORT I.S., An effective method of solving differential games with many pursuers. Dokl. AN SSSR, 256, 3, 1981.
4. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, Nauka, 1974.
5. PONTRYAGIN L.S., Linear differential games of pursuit. Matem. sb., 112, Issue 3, 1980.
6. GRIGORENKO N.L., On the linear problem of pursuit by several objects. Dokl. AN SSSR, 258, 2, 1981.
7. PSHENICHNYI B.N., Convex Analysis Extremal Problems. Moscow, Nauka, 1980.
8. PSHENICHNYI B.N., The Necessary Conditions for an Extremum. Moscow, Nauka, 1982.
9. VARGA J., Optimal Control by Differential and Functional Equations. Moscow, Nauka, 1977.

Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.4,pp.404-413,1985
Printed in Great Britain

0021-8929/85 $10.00+0.00
Pergamon Journals Ltd.

OPTIMAL CONTROL WITH A FUNCTIONAL AVERAGED ALONG THE TRAJECTORY*

A.I. PANASYUK and V.I. PANASYUK

A set of infinite optimal trajectories (IOT) is defined. It is shown that in an arbitrary fixed time interval any optimal trajectory of a system for a problem with fairly large control time (and arbitrary initial conditions) can be uniformly approximated to some IOT with the desired accuracy. Sufficient conditions are presented which ensure the existence of IOT, and the structure of the IOT set is investigated, using the rearrangement operator. The set of main trajectories is defined, and the correctness of that definition is proved. A chain of approximations is obtained: IOT approximate optimal trajectories of finite length, and the main trajectories approximate the IOT.

The properties of optimal trajectories of considerable length, and of IOT and main trajectories are investigated by solving the problem of optimal control, with a functional averaged along the trajectory. It is shown that a limit time-averaged value of the quality functional on optimal trajectories of the problems in a finite interval, when its duration increases without limit, does exist, is independent of the selection of the initial and finite conditions of these problems, and is equal to its value on any IOT. For a problem of "optimum in the mean" control the exact lower bound of the functional averaged over time does not change, if one limits the consideration only to periodic modes of the system with all possible periods. The paper continues investigations carried out in /1—4/. A somewhat different aspect of the problem of the asymptotic forms of the optimal trajectories of a control system was studied in /5, 6/, and a number of similar problems was investigated in /7—11/ etc. Generalizations to problems with discrete times were considered in /12, 13/.

1. Formulation of the problem. The following problem of optimal control is considered:

$$\frac{dx}{dt}=f(x,u),\quad u\in U\subset R^r;\quad x\in X\subset R^n \tag{1.1}$$

*Prikl.Matem.Mekhan.,49,4,524-535,1985